MA 161: Lesson 24

Graphing Functions - Part I (4.4)

Review example (concave up, concave down, increasing, decreasing)
Guidelines for sketching functions
Applying what we learnt so far to sketch functions

Coming up next: Graphing Functions - Part II.

Office Hours

M, W, F: 2:45pm - 4:15pm

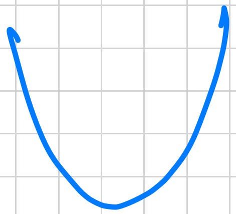
Review: What does $f'(x)$, $f''(x)$ tell us?

$f' > 0 \rightsquigarrow f \uparrow$

$f' < 0 \rightsquigarrow f \downarrow$

$f' = 0$ or DNE $\rightsquigarrow$ Critical points

$f'' > 0 \rightsquigarrow f$ is concave up



$f'' < 0 \rightsquigarrow f$ is concave down

$f'' = 0 \rightsquigarrow$ inflections

+

Domain

x-intercepts, y-intercepts

Vertical Asymptotes

End Behaviour

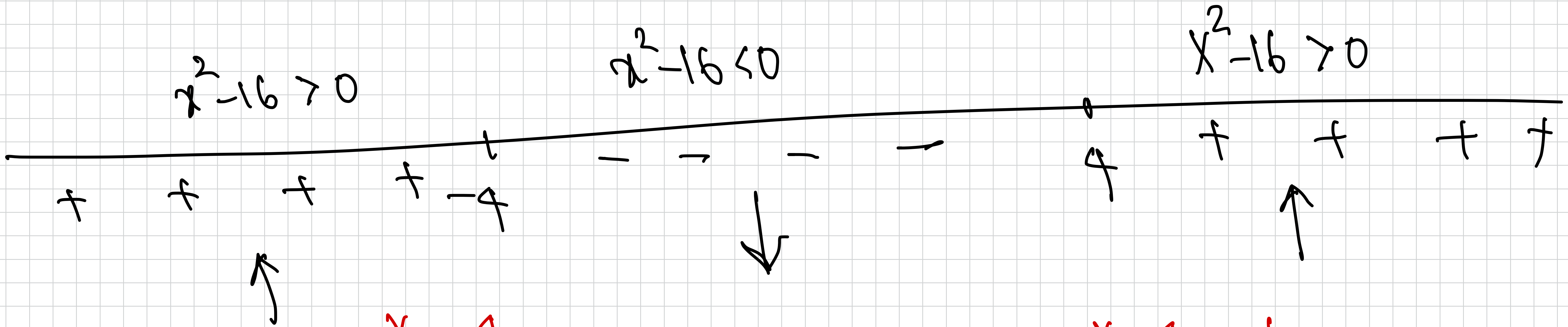
eg: $f(x) = \frac{x^3}{3} - 16x$

find intervals where the function is increasing and decreasing.

$f' > 0$ $f' < 0$

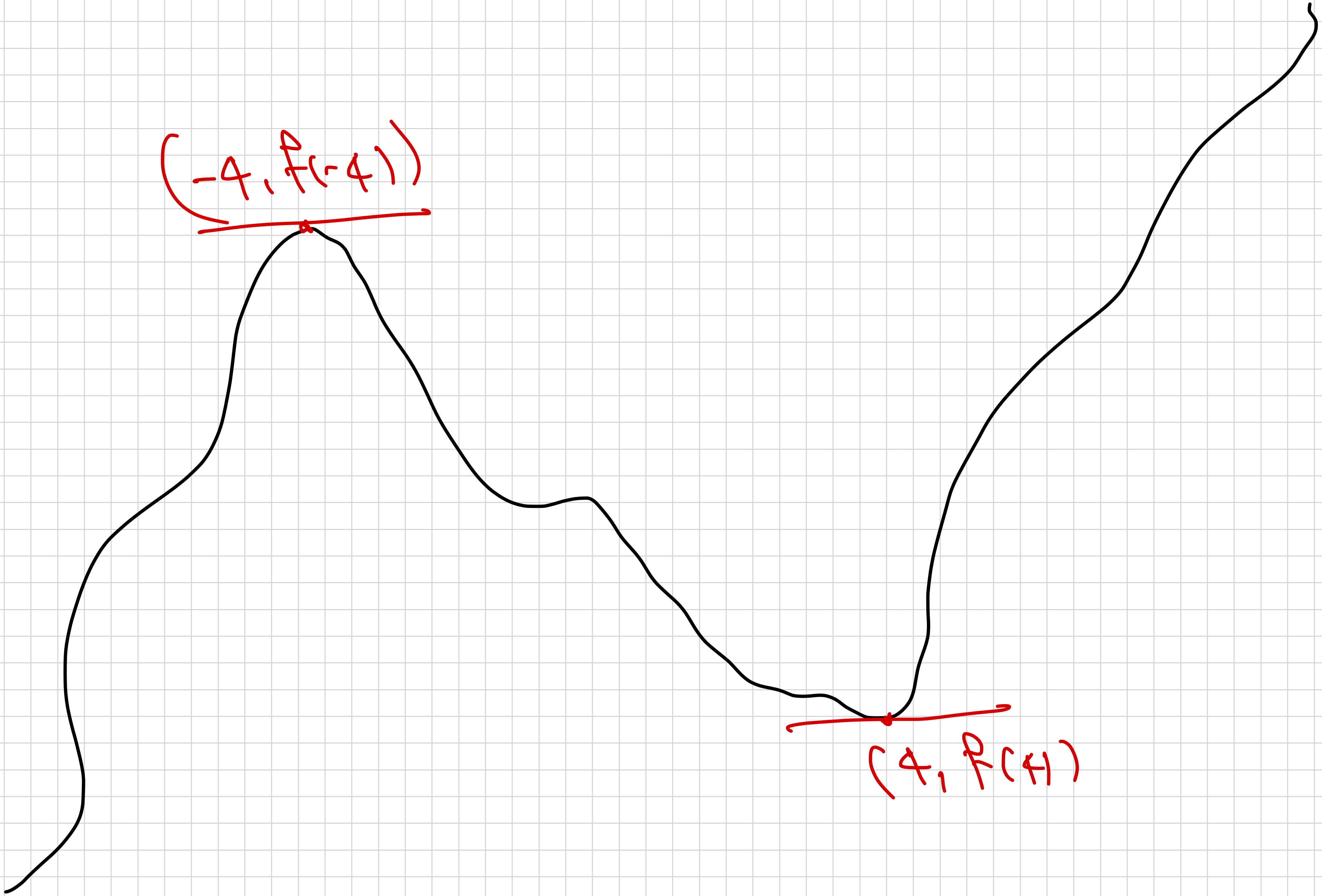
$f'(x) = x^2 - 16$

Critical points: $x^2 - 16 = 0 \to x = 4$ and $x = -4$



$X = -4$ is Local max.

$X = 4$ is Local min


(-4, f(-4))
(4, f(4))

$f(x) = \frac{x^3}{3} - 16x$

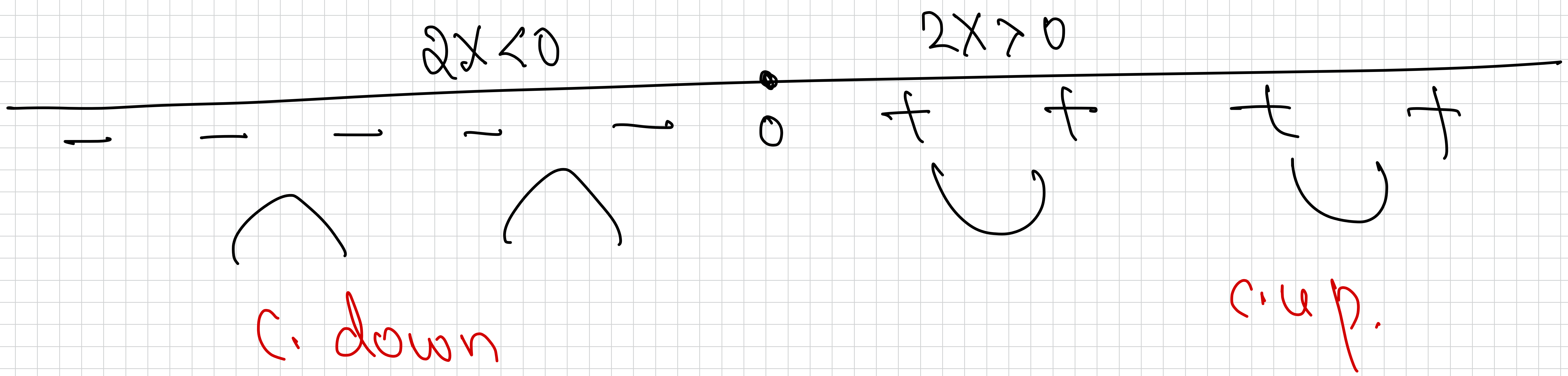
find intervals where the function is concave up and down.

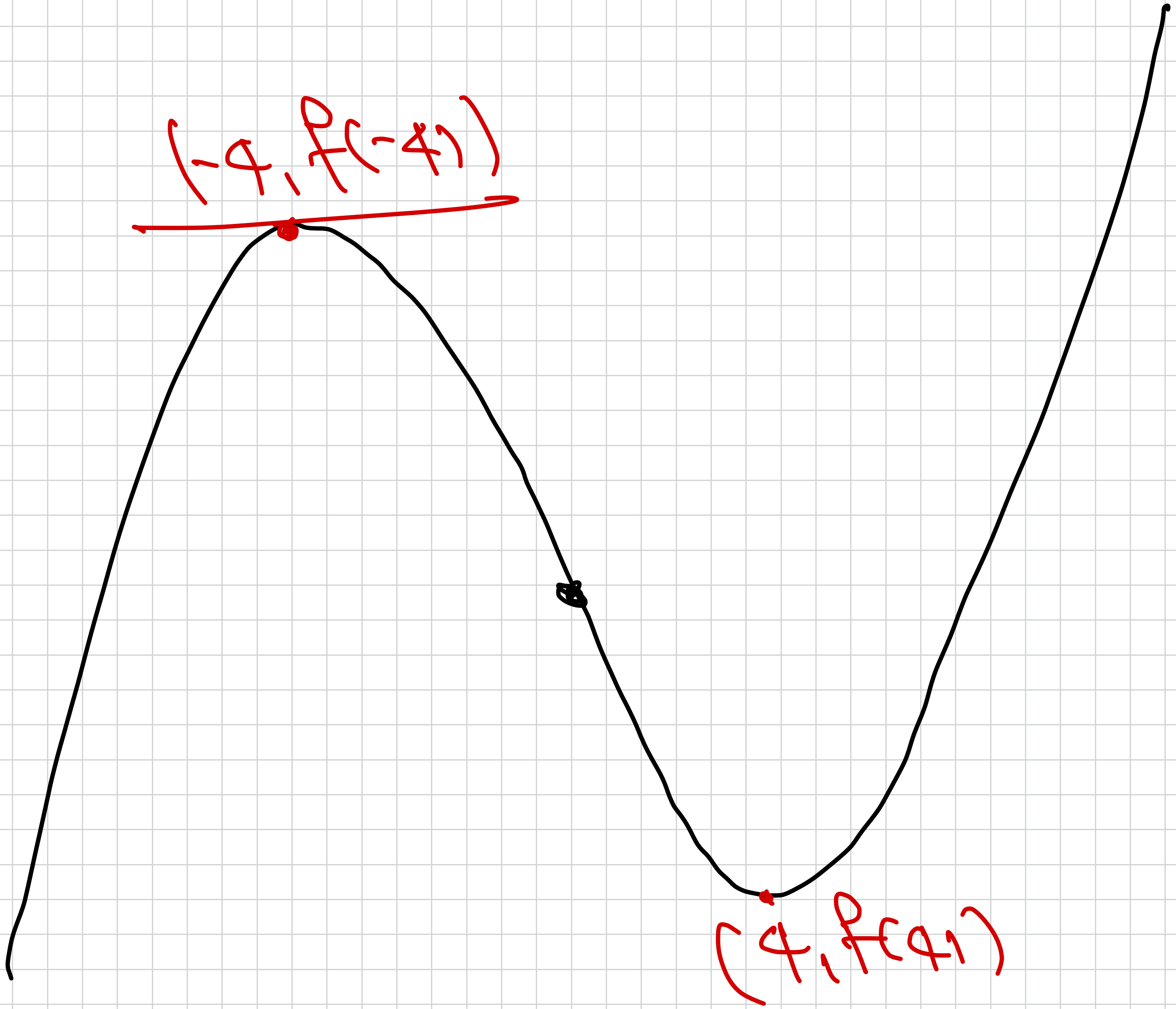
$f'' > 0$ $f'' < 0$

$f'(x) = x^2 - 16$

$f''(x) = 2x$

inflection points: $f''(x) = 2x = 0$ ↪ $x = 0$




(-4, f(-4))
(4, f(4))

More information about $f(x)=\frac{x^3}{3}-16x$

Domain: $(-\infty, \infty)$

X-intercept: $f(x)=0 \rightsquigarrow \frac{x^3}{3}-16x=0 \rightsquigarrow \frac{x}{3}\left[x^2-48\right]=0$

$x=0, \quad x=\sqrt{48}, \quad x=-\sqrt{48}$

Y-intercept: $\rightsquigarrow x=0 \Rightarrow f(0)=0.$

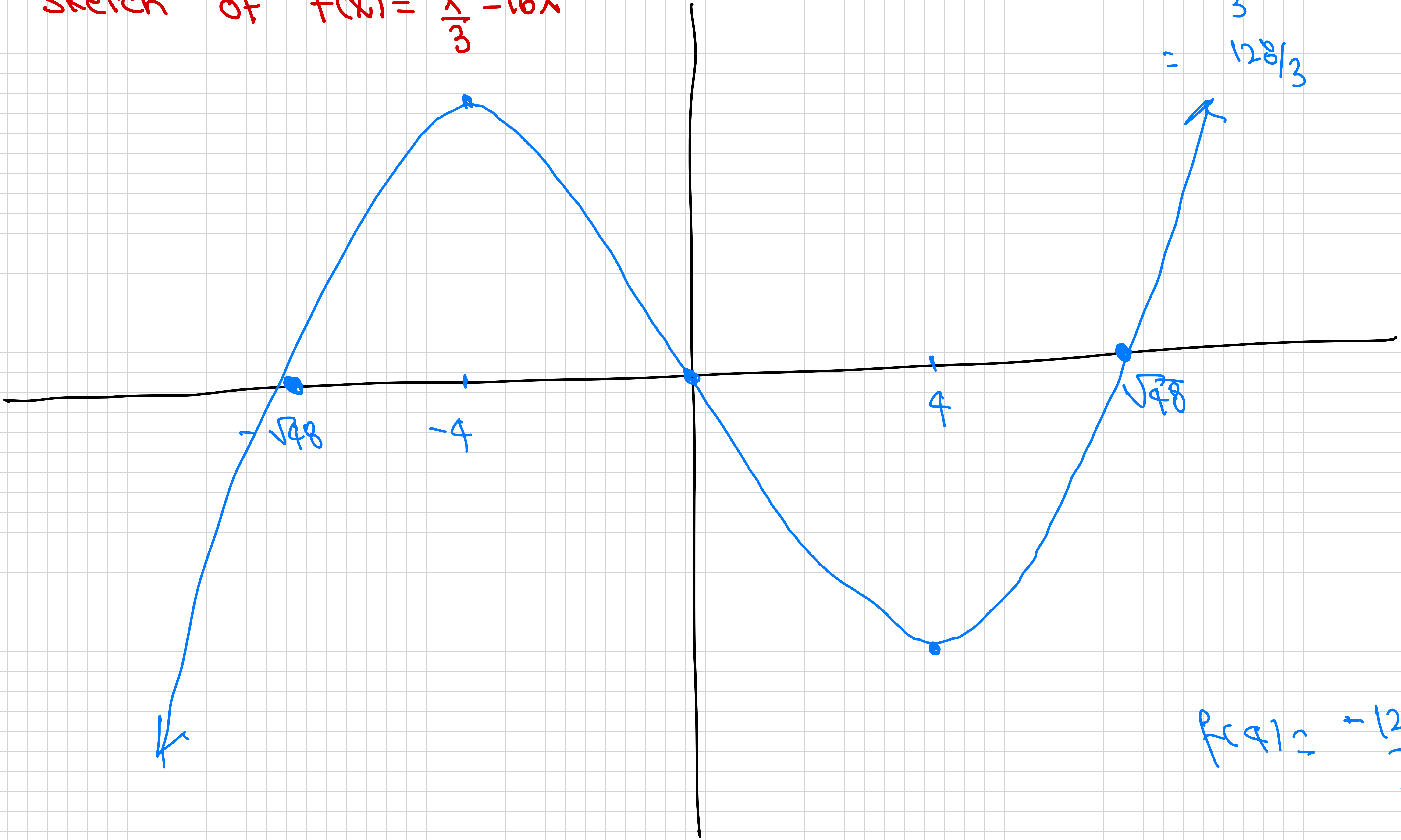
V. Asymptotes: No V.A

End Behaviour: $\lim\limits_{x\to\infty} \frac{x^3}{3}-16x=+\infty \quad , \quad \lim\limits_{x\to-\infty} \frac{x^3}{3}-16x=-\infty$

Sketch of $f(x) = \frac{x^3}{3} - 16x$

$f(-4) = -\frac{64}{3} + 64$

$= 128/3$



$f(4) = -\frac{128}{3}$

Sketching Guidelines

① Find Domain

② Find x-intercepts & y- intercepts

③ Are there any Vertical Asymptotes

④ Determine end behaviour: $\lim_{x \to \infty} f(x)$, $\lim_{x \to -\infty} f(x)$

⑤ Use f' to find intervals where $f(x)$ is increasing & decreasing

⑥ Use f'' to find intervals where $f(x)$ is Concave up & Concave down.

eg: Sketch $f(x) = \frac{x^3}{x^2-1}$
#

① **Domain:** $x \neq 1, \; x \neq -1 \rightsquigarrow (-\infty,-1) \cup (-1,1) \cup (1,\infty)$

OR

$(-\infty,\infty) \setminus \{-1,1\}$

② **x-intercepts:** $f(x) = \frac{x^3}{x^2-1} = 0 \rightsquigarrow x = 0$

y-intercept: $f(0) = \underline{\underline{0}}$

③

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{x^3}{x^2-1} \approx \frac{-1}{\text{small, -ve}} = +\infty$$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{x^3}{x^2-1} \approx \frac{-1}{\text{small, +ve}} = -\infty$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^3}{x^2-1} = \frac{1}{\text{small, +ve}} = +\infty$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^3}{x^2-1} = \frac{1}{\text{small, -ve}} = -\infty$$

④ End Behaviour:

$f(x) = \frac{p(x)}{q(x)}$ = rational function

if degree of $p(x)$ = degree of $q(x) + 1$

we have slant asymptote

When is x is Large we $x^3 - 1 \approx x^2$

$$\frac{x^3}{x^2-1} \approx \frac{x^3}{x^2} \approx x$$

$y = x$ is a slant Asymptote

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} x = +\infty$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x = -\infty$$

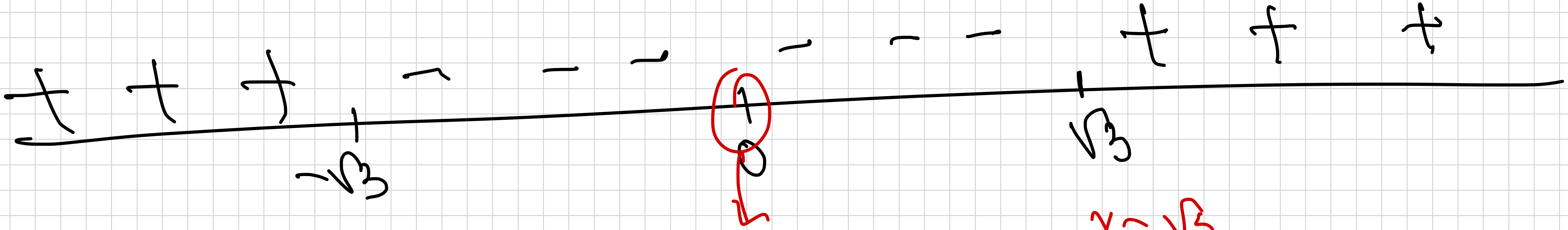
$f(x) = \frac{x^3}{x^2-1}$

$f'(x) = \frac{d}{dx}\left[\frac{x^3}{x^2-1}\right] = \frac{3x^2(x^2-1) - x^3(2x)}{(x^2-1)^2} = \frac{3x^4 - 3x^2 - 2x^4}{(x^2-1)^2}$

$= \frac{x^4 - 3x^2}{\boxed{(x^2-1)^2}} > 0$

Numerator: $x^4 - 3x^2 = x^2(x^2-3) = 0$

$\rightarrow$ $x = 0$, $x = \sqrt{3}$, $x = -\sqrt{3}$



$x = -\sqrt{3}$ is max

$x = 0$ is neither max nor min

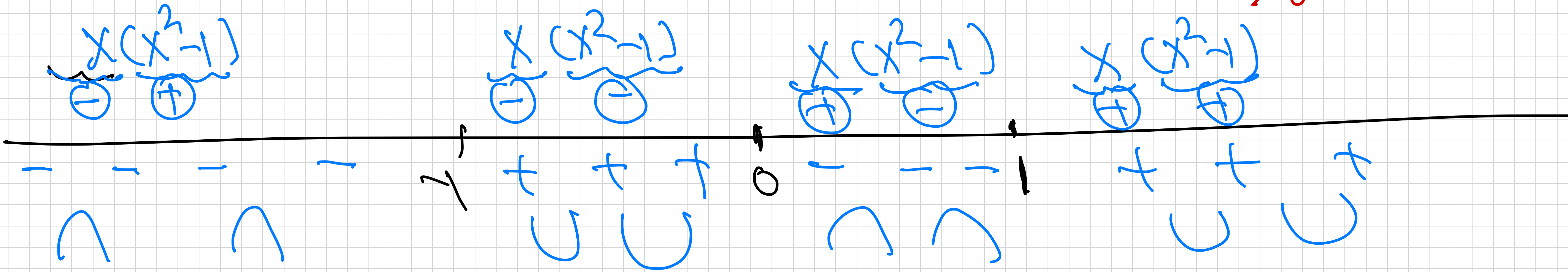
$x = \sqrt{3}$ is min

$$f'(x) = \frac{x^4-3x^2}{(x^2-1)^2}$$

$$f''(x) = \frac{(4x^3-6x)\underline{(x^2-1)}^2 - (x^4-3x^2)\left(2\underline{(x^2-1)}(2x)\right]}{(x^2-1)^4}$$

$$= \frac{(x^2-1)\left[(4x^3-6x)(x^2-1) - 4x(x^4-3x^2)\right]}{(x^2-1)^4}$$

$$= \frac{(x^2-1)\left(\cancel{4x^5} - 6x^3 - 4x^3 + 6x - \cancel{4x^5} + 12x^3\right)}{(x^4-1)^4} = \frac{(x^2-1)(x)\overset{>0}{\boxed{(x^2+6)}}}{\underset{>0}{\boxed{(x^4-1)^4}}}$$



$f(x) = \frac{x^3}{x^2-1}$